Math 245C Lecture 14 Notes

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1 Limits of Scaled Convolutions

1.1 Limits of scaled convolutions

Recall that if $\phi \in L^1(\mathbb{R}^n)$ and

$$\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right),$$

then

$$\|\phi_t\|_1 = \|\phi\|.$$

Theorem 1.1. *Let $\phi \in L^1$, let $f \in L^p$ and let $1 \leq p \leq \infty$.*

1. *If $p < \infty$,*
$$\lim_{t \to 0} \|\phi_t * f - af\|_p = 0, \qquad a = \int_{\mathbb{R}^n} \phi(y)\,dy.$$

2. *If $p = \infty$ and f is uniformly continuous, then*
$$\lim_{t \to 0} \|\phi_t * f - af\|_u = 0, \qquad a = \int_{\mathbb{R}^n} \phi(y)\,dy.$$

3. *If $O \subseteq \mathbb{R}^n$ is a bounded open set, $K \subseteq O$ is compact, and $f \in C(O) \cap L^\infty$, then*
$$\lim_{t \to 0} \|\phi_t * f - f\|_{C(K)} = 0.$$

Proof. (1) Assume $1 < p < \infty$, and set $q = p/(p-1)$. We have

$$\phi_t * f(x) - af(x) = \int_{\mathbb{R}^n} (f(x-y) - f(x))\phi_t(y)\,dy = \frac{1}{t^n}\int_{\mathbb{R}^n} (f(x-y) - f(x))\phi\left(\frac{y}{t}\right)\,dy$$

Making the change of variables, $z = y/t$, we get

$$\phi_t * f(x) - af(x) = \int_{\mathbb{R}^N} (f(x-tz) - f(x))\phi(z)\,dz.$$

So

$$\int_{\mathbb{R}^n} |\phi_t * f(x) - af(x)|^p \, dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (f(x - tz) - f(x))\phi(z) \, dz \right|^p \, dx.$$

Using Minkowski's inequality for integrals, we obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |\phi_t * f(x) - af(x)|^p \, dx \right)^{1/p} &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x - tz) - f(x)|^p |\phi(z)|^p \, dx \right)^{1/p} dz \\ &\leq \int_{\mathbb{R}^n} |\phi(z)| \|\tau_{tz} f - f\|_p \, dz \end{aligned}$$

Note that

$$\|\tau_{tz} f - f\|_p \leq 2\|f\|_p, \qquad \lim_{t \to 0} \|\tau_{tz} f - f\|_p = 0.$$

So $\|\phi_t * f - f\|_p \leq \int_{\mathbb{R}^n} \psi(t, z) \, dz$, where $|\psi| \leq 2\|f\|_p |\phi| \in L^1$. Using the dominated convergence theorem, this completes the proof of the first claim. Note that the proof also works for $p = 1$.

(2) Assume $p = \infty$, and let f be uniformly continuous. Set

$$m_f(\delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|,$$

so that

$$\lim_{\delta \to 0} m_f(\delta) = 0.$$

As we have calculated above,

$$|\phi_t * f(x) - af(x)| \leq \int_{\mathbb{R}^n} m_f(t|z|) |\phi(z)| \, dz.$$

But $m_f(t|z|)|phi(z)| \leq 2\|f\||\phi| \in L^1$. We apply the dominated convergence theorem to obtain that

$$\limsup_{t \to 0} \phi_t * f - f\|_u \leq \limsup_{t \to 0} \int_{\mathbb{R}^n} m_f(t|z|) |\phi(z)| \, dz = 0.$$

So we get the second claim.

(3) Let $2d = \operatorname{dist}(K, O^c)$. Choose a compact $K_1 \subseteq O$ such that $K \subseteq K_1$ and $\operatorname{dist}(K_1, O^c) \geq d$. Fix $\varepsilon > 0$. It suffices to show that

$$\limsup_{t \to 0} \|\phi_t * f - f\|_{C(K)} \leq \varepsilon.$$

Let $R > 0$ be large so that

$$\int_{\mathbb{R}^n \setminus B_R(0)} |\phi| \, dz < \frac{\varepsilon}{2(1 + \|f\|_\infty)}.$$

Fix $x \in K$. by our earlier calculation,

$$\phi_t * f(x) - af(x) = \underbrace{\int_{B_R(0)} (f(x-tz) - f(x))\phi(z)\,dz}_{I_1(t)} + \underbrace{\int_{\mathbb{R}^n \setminus B_R(0)} (f(x-tz) - f(x)\phi(z)\,dz}_{I_2(t)}.$$

We have

$$|I_2| \leq 2\|f\|_\infty \int_{X\setminus B_R(0)} |\phi(z)|\,dz \leq \varepsilon.$$

Since K is compact and $f \in C(K_1)$, f is uniformly continuous on K_1, and

$$\lim_{\delta\to 0} m_{K_1}(\delta) = 0, \qquad \text{where } m_{K_1}(\delta) = \sup_{\substack{|z-y|\leq\delta \\ z,y\in K_1}} |f(y) - f(z)|.$$

Since $x \in K$ if $tR < d$, then $x, x - tz \in K_1$ if $|z| < R$. So

$$|I_1| \leq \int_{B_R(0)} m_{K_1}(tR)|\phi(z)|\,dz = m_{K_1}(tR) \int_{B_R(0)} |\phi(z)|.$$

Hence, $\lim_{t\to 0} I_1(t) = 0$. So we get

$$\limsup_{t\to 0} \|\phi_t * f - f\|_{C(K)} \leq \varepsilon. \qquad \square$$

Remark 1.1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Borel function. Assume $c, \varepsilon > 0$ and

$$|\phi(z)| \leq \frac{C}{(1+|z|)^{n+\varepsilon}}$$

for all z. Note that $\phi \in L^p$ for any $p \in [1,\infty]$. Indeed, if $1 \leq p < \infty$,

$$\|\phi_p\|^p \leq c^p \int_{\mathbb{R}^n} \frac{1}{(1+|z|)^{(n+\varepsilon)p}}\,dz = C^p|S^{n-1}| \int_0^\infty \frac{r^{n-1}}{(1+r)^{(n+\varepsilon)p}}\,dr$$

There for, for any $q \in [1,\infty]$ and any $f \in L^q$, $\phi_t * f(x)$ exists.

Our goal is to show thati f x is a Lebesgue point for f, then

$$\lim_{t\to 0} \phi_t * f(x) = f(x).$$